Numerical Study of the Drag on a Fluid Sphere

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Knowledge of the basic phenomena involved between a fluid sphere and an external flow is important to the understanding of many liquid–liquid, solid–liquid, and gas–liquid systems in chemical and biochemical processing, environmental engineering, and minerals refining.

The flow structure depends on the fluid-particle Reynolds number, on the viscosity ratio ($\kappa = \mu_d/\mu_c$), and on the density ratio ($\gamma = \rho_d/\rho_c$) between the dispersed phase and the continuous phase. The flows outside and inside a fluid sphere moving with a steady-state velocity were obtained analytically by Hadamard (1911) and Rybczynski (1911) for creeping flow (low Reynolds number). At higher Reynolds number, no analytical solutions exist, and numerical solutions must be sought.

Abdel-Alim and Hamielec (1975) used a finite difference method to calculate the steady motion at $Re = 50$ and viscosity ratio $\kappa = 1.4$. This work was extended to higher Reynolds numbers (up to 200) by Rivkind and Ryskin (1976) and Rivkind et al. (1976). A correlation based on the best fitting of these numerical results was proposed to estimate the drag coefficient as a function of Reynolds number and viscosity ratio κ (Rivkind and Ryskin, 1976)

$$C_d = \frac{1}{1+\kappa}\left[\kappa\left(\frac{24}{Re} + 4Re^{-1/3}\right) + 14.9Re^{-0.78}\right] \qquad (1)$$

Oliver and Chung (1987) used a different method (a series truncation method with a cubic finite-element method) for moderate Reynolds numbers $Re = 50$. They proposed an equation that best fit their own numerical results for low but finite Reynolds numbers between 0 and 2

$$C_d = C_{d_{cf}} + 0.4\left(\frac{3\kappa + 2}{1+\kappa}\right)^2 \qquad (2)$$

where $C_{d_{cf}}$ is the drag coefficient for creeping flow [Hadamard (1911) and Rybczinski (1911)]

$$C_{d_{cf}} = \frac{8}{Re}\,\frac{3\kappa + 2}{1+\kappa} \qquad (3)$$

In this study, which is an extension of the previous works, we resolve the Navier–Stokes equations, inside and outside a fluid sphere. Based on our numerical results, a predictive equation for drag coefficients is proposed for Reynolds numbers in the range $0.01 < Re < 400$ and viscosity ratio κ from 0 to 1,000.

Governing Equations

Since the flow is considered axisymmetric, the Navier–Stokes equations can be written in terms of the stream function and vorticity (Ψ and ω) in spherical coordinates (Clift et al., 1978; Sadhal et al., 1996)

$$E^2\Psi_d = \omega_d r \sin\theta \qquad (4)$$

$$\frac{\mu_c}{\mu_d}\frac{\rho_d}{\rho_c}\frac{Re}{2}\left[\frac{\partial\Psi_d}{\partial r}\frac{\partial}{\partial\theta}\left(\frac{\omega_d}{r\sin\theta}\right) - \frac{\partial\Psi_d}{\partial\theta}\frac{\partial}{\partial r}\left(\frac{\omega_d}{r\sin\theta}\right)\right]\sin\theta = E^2(\omega_d r\sin\theta) \qquad (5)$$

where

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$$

Outside the fluid sphere, the preceding equations are still valid, but for numerical reasons discussed later the radius coordinate is transformed via $r = e^z$, where z is the logarithmic radial coordinate. The results are as follows

$$E^2\Psi_c = \omega_c r \sin\theta \qquad (6)$$

$$\frac{Re}{2}\left[\frac{\partial\Psi_c}{\partial z}\frac{\partial}{\partial\theta}\left(\frac{\omega_c}{e^z\sin\theta}\right) - \frac{\partial\Psi_c}{\partial\theta}\frac{\partial}{\partial z}\left(\frac{\omega_c}{e^z\sin\theta}\right)\right] \times e^z\sin\theta = e^{2z}E^2(\omega_c e^z\sin\theta) \qquad (7)$$

All variables are normalized by introducing the following dimensionless quantities

$$r = r'/a; \quad \omega = \omega' a/U; \quad \Psi = \Psi'/(Ua^2); \quad Re = 2aU/\nu_c;$$
$$\text{and } u = -\frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}; \quad v = \frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r}$$

where a is the fluid sphere radius, Re is the Reynolds number, U_∞ is the terminal velocity, ν_c is the kinematic viscosity, the primes denote the dimensional quantities, and subscripts

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Table 1. Present Results for C_d vs. Other Numerical Studies

	Bubble			Rigid Sphere			
Re	Brabston and Keller (1975)	Ryskin and Leal (1984)	Present Results ($\kappa=0$)	Leclair et al. (1970)	Alassar et al. (2000)	Feng and Michaelides (2000)	Present Results ($\kappa=1{,}000$)
1	17.59	17.5	17.58	27.32	—	26.98	27.55
10	2.35	2.43	2.49	4.33	4.28	—	4.42
20	1.36	1.41	1.43	2.74	2.80	2.68	2.77
30	—	—	1.03	2.13	—	—	2.15
50	—	0.67	0.69	1.86	1.84	1.77	1.81
100	—	0.38	0.39	1.10	1.08	1.09	1.08
200	0.20	0.22	0.22	0.77	—	—	0.76
400	—	—	0.14	0.55	—	—	0.55

d and c refer to dispersed and continuous phase, respectively.

The boundary conditions to be satisfied are: (1) far from the fluid sphere ($z=z_\infty$), undisturbed parallel flow is assumed: $\omega_c=0$; $\Psi_c=0.5e^{2z}\sin^2\theta$; (2) along the axis of symmetry ($\theta=0,\pi$): $\Psi_c=0$, $\omega_c=0$, $\Psi_d=0$, $\omega_d=0$; (3) across the interface ($z=0$ or $r=1$), the following relations take into account, respectively, negligible material transfer, continuity of tangential velocity, continuity of tangential stress

$$\Psi_c=0;\quad \Psi_d=0;\quad \frac{\partial\Psi_c}{\partial z}=\frac{\partial\Psi_d}{\partial r};\quad \frac{\mu_c}{\mu_d}\left(\frac{\partial^2\Psi_c}{\partial z^2}-3\frac{\partial\Psi_c}{\partial z}\right)=\left(\frac{\partial^2\Psi_d}{\partial r^2}-2\frac{\partial\Psi_d}{\partial r}\right)$$

The drag coefficient and surface pressure are obtained by integrating the appropriate component and are given below.

Stagnation pressure

$$P_0=\frac{8}{Re}\int_1^{r_\infty}\left(\frac{\partial\omega}{\partial\theta}\right)_{\theta=0^\circ}\frac{dr}{r} \tag{8}$$

Surface pressure distribution

$$P_\theta=P_0+\frac{4}{Re}\int_0^{\theta}\left(\frac{\partial\omega}{\partial r}+\omega\right)_{r=1}d\theta-(u^2)_{r=1} \tag{9}$$

Friction drag coefficient

$$C_{df}=\frac{4}{Re}\int_0^{\pi}\left(\frac{\partial\omega}{\partial\theta}+\omega\ \mathrm{cotg}\ \theta\right)_{r=1}\sin(2\theta)d\theta \tag{10}$$

Pressure drag coefficient

$$C_{dp}=\int_0^{\pi}P_\theta\sin(2\theta)d\theta \tag{11}$$

Total drag force

$$C_d=C_{dp}+C_{df} \tag{12}$$

The finite difference method is used in the present study. A grid with constant radial step, Δr, and constant angular step $\Delta\theta$ is used inside the fluid sphere. However, outside the sphere it is desirable to have a finer mesh near the sphere where the gradients are large. For this the $r=\exp(z)$ transformation is used. Then the grid in the continuous phase is generated with uniform spacing in z and θ. The stream function equations are resolved using the SOR method (successive overrelaxation); the vorticity equations are solved using the alternating direction implicit (ADI) algorithm. Details of the numerical methods used to obtain the results in this article are given by Saboni et al. (2002). However, a number of other researchers have used the ADI method in the case of flow past a rigid sphere, a bubble, or a spherical water drop (Rimon and Cheng, 1969; Ryskin and Leal, 1984; Saboni, 1991).

Results and Discussion

We limited our investigation to axisymmetric flow past a spherical particle and we consider only Reynolds number up to 400. This Reynolds number is the upper limit for which the flow is axisymmetric in the case of rigid spheres, and is also the upper limit for which an air bubble in water remains nearly spherical. We have computed solutions for Reynolds numbers in the range $1<Re<400$ and viscosity ratio κ from 0 to 1,000.

In order to verify the method of solution, the present results for the two limiting cases ($\kappa=0$ and $\kappa=1{,}000$) are compared with results from the literature. The case when $\kappa=0$ corresponds to uniform flow past a sphere with a fully mobile interface (air bubble in water), while the case when $\kappa=1{,}000$ corresponds to uniform flow past a sphere with a nearly immobile interface (rigorously, the rigid sphere corresponds to $\kappa=\infty$). In the first case, the values of the overall drag coefficient computed from our numerical analysis are compared with the results given by Brabston and Keller (1975) and Ryskin and Leal (1984). In the second case, our results

Table 2. Drag Coefficients for Fluid Sphere for Various Viscosity Ratio

Re	$\kappa=0$	$\kappa=0.3$	$\kappa=1$	$\kappa=5$	$\kappa=10$	$\kappa=100$	$\kappa=1{,}000$
1	17.58	19.93	22.6	25.9	26.65	27.46	27.55
10	2.49	2.91	3.43	4.09	4.24	4.40	4.42
20	1.43	1.71	2.07	2.54	2.65	2.75	2.77
30	1.03	1.26	1.57	1.96	2.05	2.14	2.15
50	0.69	0.86	1.10	1.44	1.51	1.58	1.59
100	0.39	0.51	0.69	0.97	1.03	1.08	1.08
200	0.22	0.31	0.44	0.69	0.74	0.76	0.76
300	0.17	0.24	0.36	0.58	0.60	0.63	0.63
400	0.14	0.21	0.33	0.53	0.54	0.55	0.55

Table 3. Correlations for Drag Coefficient for Viscosity Ratio $\kappa = 1$

Re	Eq. 1 (Rivkind and Ryskin, 1976)	Eq. 13 (Present Results)	Eq. 2 (Oliver and Chung, 1987)	Eq. 3 (H-R Creeping Flow)
0.01	1480	2003	2003	2000
0.1	169.2	202.43	202.5	200
1	21.45	22.33	22.5	20
2	11.93	12.24	12.5	
5	5.69	5.83		
10	3.37	3.42		
50	1.13	1.14		
100	0.76	0.76		
400	0.37	0.37		

are compared with those from Leclair et al. (1970), Alassar et al. (2000), and Feng and Michaelides (2000). The results of the calculations for the overall drag coefficient and the comparison with the known results are given in Table 1, which shows good agreement between our results and those from the studies just cited.

Values of the total drag coefficient, C_d, from our calculations for the range $Re = 1-400$ and $k = 0-1{,}000$ are presented in Table 2. It is observed that, at a fixed viscosity ratio, the drag coefficient decreases as the Reynolds number increases. However, for a fixed Reynolds number, the drag coefficient increases as the viscosity ratio increases, and reaches a limit value corresponding to the drag coefficient for a solid spherical particle.

A correlation of these numerical data has been obtained. This correlation, which is reduced to the Hadamard−Rybzcynski (H-R) solution for $Re \to 0$, is as follows

$$C_d = \frac{\left[\kappa\left(\frac{24}{Re} + \frac{4}{Re^{1/3}}\right) + \frac{14.9}{Re^{0.78}}\right]Re^2 + 40\frac{3\kappa + 2}{Re} + 15\kappa + 10}{(1+\kappa)(5+Re^2)} \tag{13}$$

Values of the drag coefficient from Eq. 13 are reported in Table 3 and compared to those from Rivkind and Ryskin (1976), Oliver and Chung (1987), and Hadamard−Rybczynski's correlation results for Reynolds numbers ranging between 0.01 and 400 and a viscosity ratio $\kappa = 1$. From Table 3, we can see that the results from Eq. 13 agree well with those carried out by Hadamard−Rybczynski and Oliver and Chung equations (Eqs. 2 and 3) for small Reynolds numbers. It is also seen that for $Re < 1$ the Rivkind and Ryskin equations deviate progressively from the creeping flow results. On the other hand, for large Reynolds numbers excellent agreement between our results and those of Rivkind and Ryskin (1976) is found.

The $\kappa = 1$ case is given as an example, but it is very easy to confirm these various observations for any other value of κ (by comparing the different correlations). We can, thus, suggest that Eq. 13 should be used to determine the drag coefficient for a fluid sphere for the Reynolds numbers ranging between 0.01 and 400 and for any viscosity ratio between the dispersed and the continuous phases.

Literature Cited

Abdel-Alim, A. H., and A. E. Hamilec, "A Theoretical and Experimental Investigation of the Effect of Internal Circulation on the Drag of Spherical Droplets Falling at Terminal Velocity in Liquid Media," *Ind. Eng. Chem. Fundam.*, **14**, 308 (1975).

Alassar, R. S., H. M. Badr, and R. Allayla, "Viscous Flow Over a Sphere with Fluctuations in the Free-Stream Velocity," *Comp. Mech.*, **26**, 409 (2000).

Brabston, D. C., and H. B. Keller, "Viscous Flows Past Spherical Gas Bubbles," *J. Fluid Mech.*, **69**, 179 (1975).

Clift, R., J. R. Grace, and M. E. Weber, *Bubbles, Drops and Particles*, Academic Press, New York (1978).

Feng, Z. G., and E. E. Michaelides, "A Numerical Study of the Transient Heat Transfer from a Sphere at High Reynolds and Peclet Numbers," *Int. J. Heat and Mass Transfer*, **43**, 219 (2000).

Hadamard, J., "Mouvement Permanent lent d'une Sphère Liquide et Visqueuse dans un Liquide Visqueux," *C.R. Acad. Sci.*, **152**, 1735 (1911).

Leclair, B. P., A. E. Hamielec, and H. R. Pruppacher, "A Numerical Study of the Drag on a Sphere at Intermediate Reynolds Numbers," *J. Atmos. Sci.*, **27**, 308 (1970).

Oliver, D. L. R., and J. N. C. Chung, "Flow About a Fluid Sphere at Low to Moderate Reynolds Numbers," *J. Fluid Mech.*, **177**, 1 (1987).

Rivkind, V. Y., and G. M. Ryskin, "Flow Structure in Motion of Spherical Drop in Fluid Medium at Intermediate Reynolds Numbers," *Fluid Dyn.*, **11**, 5 (1976).

Rivkind, V. Y., G. M. Ryskin, and G. A. Fishben, "Flow Around a Spherical Drop at Intermediate Reynolds Numbers," *Appl. Math. Mech.*, **40**, 687 (1976).

Rimon, Y., and S. I. Cheng, "Numerical Solution of a Uniform Flow Over a Sphere at Intermediate Reynolds Numbers," *Phys. Fluids*, **12**, 949 (1969).

Rybczynski, W., "Translatory Motion of a Fluid Sphere in a Viscous Medium," *Bull. Int. Acad. Pol. Sci. Lett., CI Sci. Nat.*, Ser. A., p. 40 (1911).

Ryskin, G., and L. G. Leal, "Numerical Solution of Free-Boundary Problems in Fluid Mechanics. Part 2. Buoyancy-Driven Motion of Gas Bubble Through a Quiescent Liquid," *J. Fluid Mech.*, **148**, 1 (1984).

Saboni, A., "Contribution à la Modélisation du Transfert de Masse Entre un Gaz et une Goutte d'eau," Thèse de Doctorat de l'Institut National Polytechnique de Toulouse, Toulouse, France (1991).

Saboni, A., S. Alexandrova, and C. Gourdon, "The Effect of the Dispersed to Continuous Phase Viscosity Ratio on Flow Past a Fluid Sphere," CD-ROM of Full Texts of the *Int. Cong. of Chemical and Process Engineering, CHISA'2002*, Prague (2002).

Sadhal, S. S., P. S. Ayyaswamy, and J. N. C. Chung, *Transport Phenomena with Drops and Bubbles*, Springer-Verlag, Berlin (1996).

Manuscript received Jan. 15, 2002, and revision received June 19, 2002.